JHEP HYPER VERSION

Received: *August 10, 2000*, Accepted: *August 30, 2000*

On the equivalence of noncommutative models in various dimensions and brane condensation

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Abstract: Here we construct a map from the algebra of fields in two-dimensional noncommutative of U(1) Yang-Mills fields interacting with Kaluza-Klein scalars to a D-dimensional one, as a solution in the two-dimensional model. This proves the equivalence of noncommutative models in various (even) dimensions. Physically this map describes condensation of D1-branes.

Keywords: M-Theory, D-branes, M(atrix) Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Noncommutative models appear to be relevant to the description of various aspects of string theory [1, 2].

An approach to noncommutative gauge theories can be developed using matrix models [3, 4] in the limit of large matrices.

Compactifications of these models to noncommutative tori was shown to yield the noncommutative Yang–Mills models [5]. The noncommutative tori correspond to matrix configurations in the IKKT model having *nondegenerate* scalar commutator in the limit $N \to \infty$. Although, at infinite matrix size these configurations solve equations of motion, they do not correspond to local minima of the classical action, and therefore do not contribute at the level of perturbation theory. However, due to their large entropical factor these configurations should become important in the strong coupling regime.

In the case of finite N one can construct a map from the matrix model to some kind of non-commutative lattice gauge model (for a recent review see [6] and references therein).

The limit $N \to \infty$ was studied by the author in refs. [7, 8], where it was shown that this limit is ambiguous at least in the perturbative approach. The ambiguity consists in the fact that depending on the background solution chosen one has in the $N \to \infty$ limit either ten-dimensional Yang-Mills-type model or its reductions to lower dimensions.

In what follows we will start with a particular choice from the variety of possible models arising in the $N \to \infty$ limit of the IKKT matrix model. We are going to show that in fact there is a wide universality among these models, in particular, matrix fluctuation around a D-dimensional (commutative) solution are completely

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equivalent to perturbations around the mentioned configuration having nondegenerate scalar commutator in $2D$ dimensions and corresponding to noncommutative tori. This equivalence become manifest due to the possibility to absorb the kinetic term in noncommutative gauge models.[1]

Also, we explicitly build a one-to-one map from one-dimensional noncommutative gauge model to the D-dimensional one using the isomorphism of Hilbert spaces. The D-dimensional model can be implemented as a solution in the one-dimensional case while the last can be obtained through the dimensional reduction of the former.

The plan of the paper is as follows. First we review some results concerning IKKT model, after that we describe the equivalence between fluctuations around commutative/noncommutative solutions, and finally find the solution which gives the map from two- to the D-dimensional noncommutative models, and discuss the implications of this map.

For the notations and background of this work we refer the reader to the ref. [7].

2. $N = \infty$ IKKT model

Start with the IKKT model at finite N. It is given by the classical euclidean action,

$$S_{\mathrm{IKKT}} = -\frac{1}{4g^2}\,\mathrm{tr}[X_\mu, X_\nu]^2 - \bar{\psi}\Gamma^\mu[X_\mu, \psi], \tag{2.1}$$

where X_μ, and ψ are scalar and spinor matrices with large size $N \to \infty$. Note that in this paper the Greek labels always run in ten dimensions, $\mu = 1, \ldots, 10$. This model possesses a number of properties such as supersymmetry, SO(10)-Lorentz and SU(N)-gauge invariances [3].

At finite N, the only classical solutions to this model are given by sets of commuting matrices $X^{(0)}_\mu$, [7],

$$[X^{(0)}_\mu, X^{(0)}_\nu] = 0\,. \tag{2.2}$$

If eigenvalues of matrices $X^{(0)}_\mu$ form a D-dimensional lattice, where D is an arbitrary integer in the range, $0 \leq D \leq 10$, then they can be expressed as functions of D independent matrices p_i, where the Latin indices run through D dimensions, $i = 1, \ldots, D$.

In the limit $N \to \infty$ matrix fluctuations around such a background are described by a D-dimensional Yang-Mills type model. In particular, when $D = 10$, and matrices $X^{(0)}_\mu$ are independent one can identify them with p_i. (In this case the sets of Greek and Latin indices coincide.) The limiting $N = \infty$ model in this case is equivalent to one given by the following action, [7],

$$S = -\int d^Dx\, d^Dl\left(\frac{1}{4}\mathcal{F}^2_{\mu\nu}(x,l) + \mathrm{i}\bar{\psi}\Gamma^\mu\nabla_\mu\psi(x,l)\right), \tag{2.3}$$

[1] I learned this possibility from ref. [9].

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where,

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu(x,l) - \partial_\nu \mathcal{A}_\mu(x,l) + g[\mathcal{A}_\mu, \mathcal{A}_\nu]_\star(x,l) \\ \nabla_\mu \psi(x,l) &= \partial_\mu \psi(x,l) - g[\mathcal{A}_\mu, \psi]_\star(x,l).\end{aligned} \tag{2.4}$$

Here derivatives are computed with respect to x, and they are given by,

$$\partial_\mu \mathcal{A}(x,l) \equiv \mathrm{i}[X^{(0)}_\mu(l), \mathcal{A}(x,l)], \tag{2.5}$$

Commutators in eqs. (2.3)–(2.5) are computed using the star product,

$$[\mathcal{A}, \mathcal{B}]_\star = \mathcal{A} \star \mathcal{B} - \mathcal{B} \star \mathcal{A}, \tag{2.6}$$

while the star product itself is defined as,

$$\mathcal{A} \star \mathcal{B}(x,l) = e^{-\frac{\mathrm{i}}{2}\left(\frac{\partial^2}{\partial x' \partial l} - \frac{\partial^2}{\partial x \partial l'}\right)} \mathcal{A}(x,l)\mathcal{B}(x',l')\Big|_{\substack{x'=x \\ l'=l}} . \tag{2.7}$$

Here l_i (the same as l_μ) parameterise the spectrum of p_i, while x^i are coordinates on the (Fourier) dual space [7].

In the case when $D < 10$ the model is given by the reduction of (2.3) to D dimensions. In this case, x^i and l_i are D-dimensional and the set of Greek indices comes split in two subsets formed by Latin indices $i, j, k \dots$ and other one of Kaluza-Klein multiplet indices which appear upon reduction to D dimensions from 10 dimensions. We will not introduce the last type of indices, because along this paper we do not need the this split explicitly, but keep instead the Greek letters for both space-time and Kaluza-Klein multiplets. In this case the derivative terms generalise according to (2.5), and the dimensional reduced action keeps the same form as given by eqs. (2.3)–(2.7), but one should keep in mind that $\mathcal{A}_\mu(x,l)$ in this case denote both the D-dimensional gauge field and the Kaluza-Klein scalars.

Using the Moyal correspondence, the algebra of (x,l)-functions supplied with the star product (2.6) can be seen as a representation of the D-dimensional Heiseberg algebra with the Weyl ordering prescription which is generated by l_i and x^i, acting on the Hilbert space $\mathcal{H}_D$ and satisfying the commutation relation,

$$[l_i, x^j] = -\mathrm{i}\delta^j_i\,. \tag{2.8}$$

In this case the integration over $d^D x\, d^D l$ is equivalent to taking the trace over $\mathcal{H}_D$. In what follows we will not distinguish between these two forms.

By a redefinition of field $\mathcal{A}_\mu$,

$$\mathcal{A}_\mu \to X^{(0)}_\mu(l_i) + \mathcal{A}_\mu\,, \tag{2.9}$$

one can absorb the kinetic term in eq. (2.3). As a result one has the model described by the action (in Heisenberg form),

$$S = -\operatorname{tr}_{\mathcal{H}_D}\left(\frac{1}{4}[\mathcal{A}_\mu, \mathcal{A}_\nu]^2 + \bar{\psi}\Gamma^\mu[\mathcal{A}_\mu, \psi]\right), \tag{2.10}$$

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where $\mathcal{A}$ and ψ are hermitian operators of D-dimensional Heisenberg algebra which act on $\mathcal{H}_D$ and are represented by noncommutative functions on (l_i, x^i), the trace $\mathrm{tr}_{\mathcal{H}_D}(\cdot)$ denotes the integration over noncommutative phase space, $\int d^D x d^D l(\cdot)$. Once again note that $\mu, \nu, \ldots = 1, \ldots, 10$, while the fields are defined on $2D$-dimensional noncommutative space generated by l and x.

Let us note that in the form (2.10) the model is manifestly invariant with respect to reparameterisations of (l, x) preserving the commutator (2.8).

Although, the action (2.10) or its dimensional reductions are obtained as a continuum limit of fluctuations around a commutative background ($[X_\mu^{(0)}, X_\nu^{(0)}] = 0$), one can show that the same model can be obtained as a continuum limit of fluctuations around a configuration with $[X_\mu^{(0)}, X_\nu^{(0)}] = \mathrm{i}B_{\mu\nu} \neq 0$, in $2D$ dimensions. In other words, the model (2.10) is equivalent to the U(1) non-commutative Yang-Millls model in $2D$ dimensions [5], when $2D = 10$, or its $2D$ dimensional reduction when $2D < 10$.

Indeed, consider action (2.10) in the case of $D = 5$. After shifting back the fields $\mathcal{A}_\mu \mapsto (\mathcal{A}_\mu - \tilde{l}_\mu)$, where $\tilde{l}_\mu$ are given by $\tilde{l}_i \equiv l_i$ for $\mu = i = 1, \ldots, 5$ and $\tilde{l}_{5+i} = x_i$, for $\mu = 5 + i = 6, \ldots, 10$ the action (2.10) becomes one of the 10-dimensional noncommutative U(1) Yang-Mills model.[2] The same trick can be made for any $D \leq 10$, however, the meaning of the model when $20 \geq 2D > 10$ is not yet clear, but as we are going to demonstrate later this is not a problem since any $D > 1$ model is equivalent to the $D = 1$.

3. From one dimension up to $D > 1$

In this section we consider the model given by the action (2.10) with $D = 1$, which means that the fields are defined as functions on the one-dimensional Heisenberg algebra generated by l and x, satisfying usual commutation relation,

$$[l, x] = -\mathrm{i}\hbar\,, \tag{3.1}$$

which is the same as two-dimensional noncommutative space. Let us note that the one-dimensional solution is one having the largest entropical factor [7] in the IKKT matrix model.

For convenience consider the Heisenberg algebra to be defined on a circle: $x + 2\pi L \sim x$. In this case the momentum operator l have discrete spectrum. Its eigenvalues are given by n/L, $n = 0, \pm 1, \pm 2, \ldots$ Later one can take the limit $L \to \infty$.

Consider the equations of motion for this model. Vacuum solutions (with $S = 0$), are given by the commutative sets of operators $\mathcal{A}_\mu^{(0)}$. In terms of "smooth" functions

[2] Or, oppositely, absorbing the kinetic term, one can bring the ten dimensional noncommutative U(1) Yang-Mills model to the form (2.10). This correspondence is possible since D-dimensional Heisenberg algebra coincides with $2D$-dimensional noncommutative space.

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and up to a gauge transformation, they are given by the arbitrary functions $\mathcal{A}_\mu^{(0)}(l)$. As it is generally known, two and more *continuous* functions of one variable always form a functional dependent set, therefore, the set of $\mathcal{A}_\mu^{(0)}(l)$ is a such one if $\mathcal{A}_\mu^{(0)}(l)$ are continuous functions of l. This property, however does not hold for the discontinuous functions.

After the kinetic term absorption the equations of motion are no more differential equations, thus the condition imposed one the solutions of equations of motion to be smooth functions are no more justified and can be given up. Moreover, the spectrum of l itself is discrete.

From the other hand, since the smoothness properties are strictly related to the notion of topology of the space-time, the non-smooth solution can be interpreted as the changing of the space-time topology. The space-time topology can be extracted from the set of operators $\mathcal{A}_\mu^{(0)}$ in the framework of Connes' approach [10].

Consider that the solution $\mathcal{A}_\mu^{(0)}$ now carries the attributes of the D dimensional space, i.e. they form a D-dimensional lattice, and can be expressed as functions of the basic set of independent operators l_i, $i = 1, \ldots, D$.

In what follows let us construct an explicit solution with functionally independent (and, therefore, discontinuous) $l_i(l)$.

Since eigenvalues of l and l_i form, respectively, one-dimensional and D dimensional lattices, the solution $l_i(l)$ is given explicitly by the map from the one-dimensional lattice Γ_1 of the eigenvalues of l to the D-dimensional lattice Γ_D of eigenvalues of l_i. Due to the reparameterisation invariance one can take both lattices to be regular and rectangular ones.

The map can be constructed through the following sequence of steps (the particular case for $D = 2$ is depicted in the figure (1)).

1. Map the origin of Γ_1 to the origin of Γ_D.

2. Map D points next to origin in the positive direction and D ones in negative direction of Γ_1 to the nearest neighbor points to the origin of Γ_D, using e.g. lexicographic ordering.

3. Fill the the remaining points of the D-dimensional hypercube of two lattice unites size centered at origin of Γ_D by images of $n = \pm(D+1), \pm(D+2), \ldots$ points of Γ_1.

4. In the same manner fill the next hyper-cubic shell of Γ_D by images of of points of Γ_1, etc.

5. Thus, $(2n+1)^D$ points around the origin of Γ_1 fill the hyper-cube of the size $2n$ centered at the origin of Γ_D.

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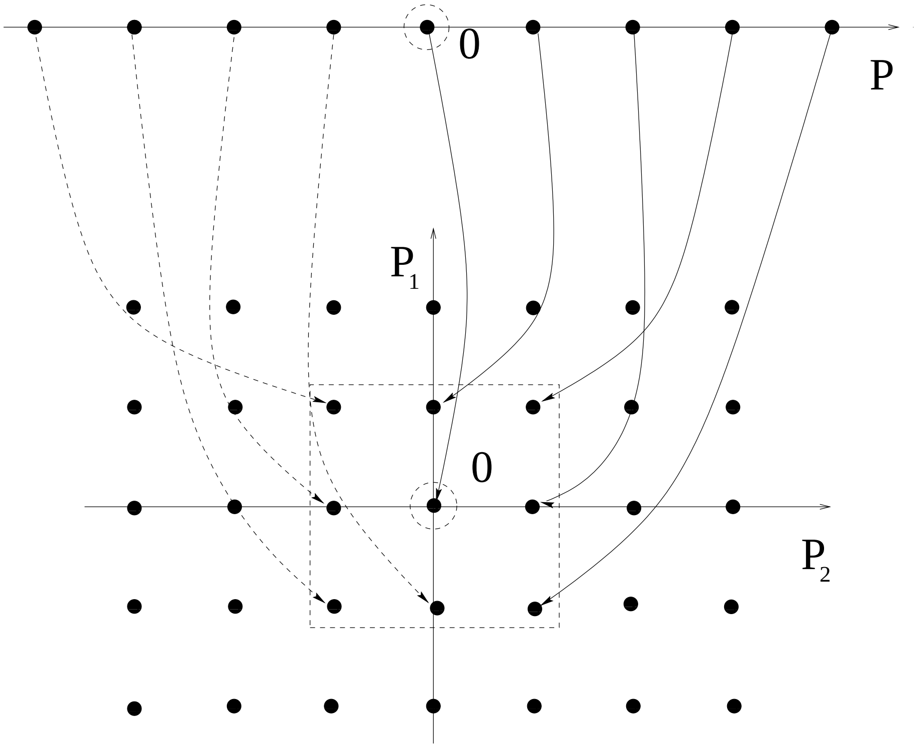

Figure 1: The isomorphic map from 1-dimensional lattice to the 2-dimensional one. It is shown how the first hyper-cubic shell (in the dotted box) of Γ_2 is filled by the points of Γ_1.

As it can be seen by the construction the map is one-to-one. However, the resulting operators l_i are functionally independent. Under this map small momenta are transfered to small momenta, and large ones to large ones. This means that respectively low/high energy states of one model are mapped into low/high energy sector of another one.

This correspondence allows one to pass from the one dimensional Heisenberg algebra to the Heisenberg algebra in arbitrary dimension $D > 1$ (or from two-dimensional noncommutative space to $2D$ dimensional one).

Indeed, using the approach of ref. [8] one can introduce for each operator l_i its canonical conjugate x^i, satisfying the Heisenberg commutation relation in D dimensions,

$$[l_i, x^j] = -\mathrm{i}\delta_i^j. \tag{3.2}$$

One can, therefore, pass from (2.10) with $D = 1$ to the equivalent description as a model in a different dimension $D > 1$.

Let us shortly describe the construction [7, 11]. Consider the eigenvalue problem for adjoint operators $P_i = [l_i, \cdot]$ and $Q^i = [x^i, \cdot]$. This problem is consistent since P_i and Q^j are commutative and self-adjoint on the space of square integrable operators with bounded P_i. The eigenvalue problem is solved by the (eigen)operators,

$$E(k, z) = e^{\mathrm{i}k_i q^i + \mathrm{i}l_i z^i}, \tag{3.3}$$

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where k_i and z^i are eigenvalues of P_i and Q^i respectively. Since, by the construction, both (l, x) and (l_i, x^i) can be represented on the same Hilbert space $\mathcal{H}_1$, (in fact, we introduced isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_D$), one can expand an arbitrary square integrable operator $\mathcal{A}(l, x)$ of original model in the $E(k, z)$ basis, using the trace over the Hilbert space of the one-dimensional Heisenberg algebra, and get an operator $\mathcal{A}(l_i, x^i)$ in the D-dimensional model,

$$\mathcal{A}(l_i, x^i) = \sum_{k_i, z^i} \tilde{\mathcal{A}}(k_i, z^i) E(k_i, z^i), \tag{3.4}$$

where,

$$\tilde{\mathcal{A}}(k_i, z^i) = \frac{1}{(2\pi)^D} \operatorname{tr}_{\mathcal{H}_1} E^\dagger(k_i, z^i) \mathcal{A}(l, x). \tag{3.5}$$

Applying this procedure to the fields in model (2.10) with $D = 1$, one obtains the equivalent description in terms of a D dimensional model with any $D > 1$.

Formally, we have introduced here a noncommutative (and discontinuous) change of variables. Indeed, since l_i and x^i are invertible functions of one-dimensional l and x, one can find their inverse, $l = l(l_i, x^i)$ and $x = x(l_i, x^i)$, and plough this dependence in the one-dimensional operator $\mathcal{A}(l, x)$ to get the function $\mathcal{A}(l_i, x^i) = \mathcal{A}(l(l_i, x^i), x(l_i, x^i))$ which is a D dimensional operator.

Now we can give the following physical implication of this construction.

The D-dimensional noncommutative gauge model describes, in fact, the a Dp-brane where $p = D - 1$ [2]. We constructed a solution in the two-dimensional noncommutative gauge model which has the meaning of $2D$ dimensional space. This solution gives the correspondence between the gauge models (gauge fields interacting with Kaluza-Klein scalars) in various even dimensions. Taking into account the brane interpretation of the noncommutative gauge models with scalar fields, this describes the condensation of D1-branes to a Dp-brane, where $p = 2D - 1$. From this point of view the multiple vacua of the IKKT model ref. [7], are nothing else than the condensation of D(-1)-branes described by the IKKT matrices to an arbitrary IIB brane (i.e. a brane with even-dimensional world-sheet), which is the $2D$-dimensional reduction of the ten-dimensional noncommutative U(1) gauge model.

4. Conclusions

Let us briefly summarise the results of this note.

First, we have shown that the model desribing the continuum limit of fluctuations of the IKKT matrix model around a commutative background, $[p_i, p_j] = 0$, in D dimensions is equivalent to one describing fluctuations around a $2D$ dimensional background satisfying, $[p_I, p_J] = \mathrm{i}B_{IJ}$, where B_{IJ} is a nondegenerate antisymmetric scalar matrix. “Physically” this means that in the singular limit $B \to 0$ corresponds to doubling of the dimensionality of the noncommutative gauge model.

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Second, we demonstrated that the noncommutative model in 2 dimensions can be isomorphically mapped to the $2D$-dimensional one. The last feature may be interpreted as the noncommutative geometry counterpart of the duality relating various branes or their condensation, [4, 12, 13, 14].

In early works on IKKT model [3], it was conjectured that this in the limit $N \to \infty$ generates the space-time as the set of expectation values of operators X_μ. In this context it seems natural that the topology and, in particular, the space-time dimensionality are also generated by the solution $l_i(l)$.

Acknowledgments

I benefited from useful discussions with many people. In particular, I would acknowledge ones with E. Kiritsis and Yu. Makeenko.

This work was made under auspices of FSA program: ICTP-BLTP with additional support from the High Energy Section of ICTP, it is a pleasure to thank its director Prof. S. Randjbar-Daemi.

This work was supported by RFBR grant no.99-01-00190, INTAS grant no. 950681, Scientific School Support grant no.96-15-0628.

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